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An Inequality Involving Powers of Sums of Powers

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A generalized version is proved of the following inequality, arising in a study of invertible measure preserving transformations: $(\sum_{i=1}^N x_i^n)^{1/n}(\sum_{i=1}^N x_i^m)^{1/m} \leqslant (\sum_{i=1}^N x^{mn})^{1/mn}(\sum_{i=1}^N x_i)$, where $x_i \geqslant 0$, $i = 1, 2,\ldots, N$, and $(m-1)(n-1) > 0$.

1. Introduction

A simple class of invertible measure preserving transformations (metric automorphisms) defined on [0, 1] with Lebesgue measure can be defined by permuting the digits in the N-ary expansions of points in [0, 1]. In determining the equivalence classes under measure preserving isomorphisms of automorphisms of this type, [1], relationships involving powers of sums of powers of probability distributions were encountered. In particular it was necessary to know whether there is a probability distribution $\{p_i: 1 \leqslant i \leqslant N\}$, $p_i > 0$, which satisfies the polynomial equation

$$\left(\sum_{i=1}^N p_i^n\right)^m \left(\sum_{i=1}^N p_i^m\right)^n = \sum_{i=1}^N p_i^{nm}.$$

In case $N = 2$, the following elementary proof shows that no such distribution exists and since the uniform distribution $(\frac{1}{2}, \frac{1}{2})$ makes the left side smaller than the right, the inequality

$$(p_1^n + p_2^n)^m (p_1^m + p_2^m)^n < p_1^{nm} + p_2^{nm}, \tag{1.1}$$

with $p_1 + p_2 = 1$, $(m-1)(n-1) > 0$, is obtained.

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In this note a more general result is obtained which has as a corollary the inequality

$$\left(\sum_{i=1}^{N} x_i^{n}\right)^{1/n} \left(\sum_{i=1}^{N} x_i^{m}\right)^{1/m} < \left(\sum_{i=1}^{N} x_i^{nm}\right)^{1/nm} \left(\sum_{i=1}^{N} x_i\right). \tag{1.2}$$

2. Proof for $N = 2$

It is easy to see that (1.1) will follow if the polynomial

$$P(x) = (x^{mn} + 1)(x + 1)^{mn} - (x^m + 1)^n (x^n + 1)^m$$

has no positive zeros. Since $P(x^{-1}) = x^{-2mn}P(x)$, if $0 < x < 1$ is a zero of $P(x)$ then $1/x$ is also a zero and it is sufficient to show that 1 is an upper bound on the positive zeros of $P(x)$.

From a classical result in theory of equations (cf. [2, pp. 70–72]), $P(x)$ will have 1 as an upper bound for its positive zeros if the cumulative sums of the coefficients of $P(x)$ are non-negative.

Let

$$\begin{aligned} Q(x) &= (x^{mn} + 1)(x + 1)^{mn} \\ &= \sum_{l=0}^{mn} \binom{mn}{mn - l} x^{2mn-l} + \sum_{l=mn}^{2mn} \binom{mn}{l - mn} x^{2mn-l} \\ &= \sum_{l=0}^{2mn} b_l x^{2mn-l}, \end{aligned}$$

$$R(x) = (x^m + 1)^n (x^n + 1)^m = \sum_{l=0}^{2mn} c_l x^{2mn-l},$$

and

$$P(x) = Q(x) - R(x) = \sum_{l=0}^{2mn} a_l x^{2mn-l}.$$

If $m + n \leqslant t \leqslant 2mn$,

$$\begin{aligned} \sum_{l=0}^{t} a_l = \sum_{l=0}^{t} b_l - \sum_{l=0}^{t} c_l &\geqslant \sum_{l=0}^{m+n} \binom{mn}{l} - R(1) \\ &= \sum_{l=0}^{m+n} \binom{mn}{l} - 2^{m+n} = \sum_{l=0}^{m+n} \left\{\binom{mn}{l} - \binom{m+n}{l}\right\} \geqslant 0. \end{aligned}$$

If $0 < t < m + n$, write

$$R(x) = \sum_{k=0}^{n} \binom{n}{k} x^{2mn-km} + mx^{2mn-n} + R_1(x),$$

where the degree of $R_1(x)$ is $2mn - m - n$ and the coefficients of this polynomial need not be considered in the cumulative sums. Then for $0 \leqslant k \leqslant m + n$, the coefficients a_k of $P(x)$ are of the form $\binom{mn}{ml} - \binom{n}{l}$, $\binom{mn}{l}$, or $\binom{mn}{n} - m$ and since each of these is nonnegative the sums $\sum_{k=0}^{t} a_k$ are nonnegative.

3. The General Inequality

Let $(\Omega, \mathscr{A}, \mu)$ be a totally finite measure space and k a measurable function on Ω such that $0 < k \leqslant 1$ and $\mu k^{-1}\{1\} \geqslant 1$. Define the functions $L(t)$, $l(t)$, and $h(t)$ for $t > 0$ by the equations

$$L(t) = \int_\Omega k^t \, d\mu,$$

$$l(t) = \log L(t),$$

and

$$h(t) = tl(t^{-1}).$$

Lemma. *The function h is a nondecreasing convex function on* $(0, \infty)$.

Proof. By differentiation

$$h'(t) = l(t^{-1}) - t^{-1}l'(t^{-1}).$$

However,

$$vl'(v) = vL(v)^{-1} \int_\Omega k^v(\log k) \, d\mu$$

and since $\log k \leqslant 0$, $vl'(v) \leqslant 0$. Also,

$$l(v) = \log \int_\Omega k^v \, d\mu$$

$$\geqslant \log(\mu k^{-1}\{1\})$$

so that $h'(t) \geqslant 0$ for all $t > 0$.

Since $h''(t) = t^{-3}l''(t^{-1})$ and $l(t)$ is convex by the Schwarz inequality we have $h''(t) \geqslant 0$ for all $t > 0$. ▮

Theorem. *If $g(u) = h(e^{-u})$ then g is a nonincreasing convex function on* $(-\infty, \infty)$.

Proof. By the lemma $g'(u) = -e^{-u}h'(e^{-u}) \leqslant 0$ and $g''(u) = e^{-u}h'(e^{-u}) + e^{2u}h''(e^{-u}) \geqslant 0$. ▮

COROLLARY 1. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a totally finite measure space and* f *a measurable function on* Ω *such that* $0 < f \leqslant M$ *and* $\mu f^{-1}\{M\} \geqslant 1$. *If* $0 < a \leqslant b$ *and* $\lambda \geqslant 1$ *then*

$$\left\{\int_\Omega f^a\, d\mu\right\}^{1/a} \left\{\int_\Omega f^{\lambda b}\, d\mu\right\}^{1/\lambda} \geqslant \left\{\int_\Omega f^{\lambda a}\, d\mu\right\}^{1/\lambda a} \left\{\int_\Omega f^b\, d\mu\right\}^{1/b}. \qquad (3.1)$$

Proof. Define $k = f/M$. By the theorem g' is nondecreasing. Take $u = \log b$, $v = \log a$ and $h = \log \lambda$. Then

$$\int_0^h \{g'(u + t) - g'(v + t)\}\, dt \geqslant 0.$$

Integration gives the result. ∎

COROLLARY 2. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a totally finite measure space and* f *a measurable function on* Ω *such that* $0 < f \leqslant M$ *and* $\mu f^{-1}\{M\} \geqslant 1$. *For every* $c > 0$,

$$\left\{\int_\Omega f^t\, d\mu\right\}^{1/t} \left\{\int_\Omega f^{c/t}\, d\mu\right\}^{t/c}$$

is nonincreasing for $0 < t < c^{1/2}$ *and nondecreasing for* $t > c^{1/2}$.

Proof. Let $a \leqslant b$ be numbers selected from $(0, c^{1/2}]$ and take $\lambda = c/ab$. Inequality (3.1) gives

$$\left\{\int_\Omega f^a\, d\mu\right\}^{1/a} \left\{\int_\Omega f^{c/a}\, d\mu\right\}^{a/c} \geqslant \left\{\int_\Omega f^b\, d\mu\right\}^{1/b} \left\{\int_\Omega f^{c/b}\, d\mu\right\}^{b/c}.$$

If $r, s \in [c^{1/2}, \infty)$ with $r \leqslant s$, take $a = c/s$, $b = c/r$ and $\lambda = c/ab = rs/c$. Then $0 < a \leqslant b$ and $\lambda \geqslant 1$ and using these values of a, b, λ in equality (3.1) gives the second part of the statement. ∎

COROLLARY 3. *Let* $x_0 > x_1 > \cdots \geqslant 0$, $\mu_0 \geqslant 1$, $\mu_i \geqslant 0$ *for* $i = 1, 2, 3,\ldots$, $p > 0$ *and* $q > 0$. *Then*

$$\left(\sum x_i\mu_i\right)\left(\sum x_i^{pq}\mu_i\right)^{1/pq} - \left(\sum x_i^p\mu_i\right)^{1/p}\left(\sum x_i^q\mu_i\right)^{1/q}$$

is nonnegative if $(p - 1)(q - 1) \geqslant 0$ *and nonpositive if* $(p - 1)(q - 1) \leqslant 0$.

Proof. Let $f(t) = x_i + (x_{i+1} - x_i)(t - i)$ for $i \leqslant t \leqslant i + 1$, $i = 0, 1, 2,\ldots$. Let μ be the measure on the nonnegative reals consisting of point masses μ_i at i for $i = 0, 1, 2,\ldots$. Then (3.1) becomes

$$\left(\sum x_i^a\mu_i\right)^{1/a}\left(\sum x_i^{\lambda b}\mu_i\right)^{1/\lambda b} \geqslant \left(\sum x_i^{\lambda a}\mu_i\right)^{1/\lambda a}\left(\sum x_i^b\mu_i\right)^{1/b}$$

for $0 < a \leqslant b$, $\lambda \geqslant 1$. The corollary follows from this by taking the following values for a, b and λ:

If $1 \leqslant p \wedge q$ take $a = 1$, $b = p$, $\lambda = q$.

If $0 < p \leqslant 1 < q$ take $a = p$, $b = pq$, $\lambda = 1/p$.

If $0 < p < q < 1$ take $a = pq$, $b = p$, $\lambda = 1/p$. ∎

Remarks. The condition $\mu_1 \geqslant 1$ cannot be deleted from Corollary 3. Indeed, the conclusion obtains when $n = x_1 = 1$ if and only if $\mu_1 \geqslant 1$.

Corollary 3 may also be proven from Corollary 2 by using f and μ as defined in the proof of Corollary 3 and taking c in Corollary 2 to be pq.

In case $\mu(\Omega) > 1$, $h'(t) > 0$ for all $t > 0$ and consequently $g''(u) > 0$ for all $u > 0$. Thus with this extra hypothesis inequality (3.1) becomes strict. If we assume in Corollary 3 that $\mu_i > 0$ for some $i > 0$ then we have

$$\left(\sum_1^\infty x_i\mu_i\right)\left(\sum_1^\infty x_i^{pq}\mu_i\right)^{1/pq} - \left(\sum x_i^p\mu_i\right)^{1/p}\left(\sum x_i^q\mu_i\right)^{1/q}$$

is negative if $(p-1)(q-1) < 0$ and positive if $(p-1)(q-1) > 0$.

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